Problem Set 3 due September 25, at 10 AM, on Gradescope (via Stellar)

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue

Problem 1:

(1) Given numbers a and b, for which number c does the system:

$$\begin{bmatrix} 1 & -2 \\ 0 & -1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

have a solution v_1, v_2. *(5 points)*

(2) Draw the set of vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfing the conditions in part (1) on a picture of $\mathbb{R}^3$. *(5 points)*

(3) Construct a 3×4 matrix whose column space is generated by $\begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$. *(5 points)*

Solution:
(1) We may row reduce the augmented matrix

$$\begin{bmatrix} 1 & -2 & a \\ 0 & -1 & b \\ 3 & 2 & c \end{bmatrix} \xrightarrow{r_3 - 3r_1} \begin{bmatrix} 1 & -2 & a \\ 0 & -1 & b \\ 0 & 8 & c - 3a \end{bmatrix} \xrightarrow{r_3 + 8r_2} \begin{bmatrix} 1 & -2 & a \\ 0 & -1 & b \\ 0 & 0 & c - 3a + 8b \end{bmatrix}.$$

From this we learn that the original system of linear equations is equivalent to the system

$$v_1 + 2v_2 = a, \text{ and}$$

$$-v_2 = b, \text{ and}$$

$$0 = c - 3a + 8b.$$

This system has a solution if and only if $c = 3a - 8b$.

Grading Rubric: 3 points for an attempt to simplify the system by row reduction or substitution and obtain a single linear relation. 5 points if no calculation errors are made.

(2) Students should draw a plane in abc space with clearly labeled axes. The plane should be perpendicular/orthogonal to the vector $(-3, 8, 1)$.

Grading Rubric: 2 points for recognizing that the answer is a plane. 3 points for drawing any plane passing through the origin. 5 points if the plane matches whatever answer was obtained in part (1).

(3) An example is given by

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ 0 & -1 & 0 & -1 \\ 3 & 2 & 6 & 5 \end{bmatrix}.$$

It is important to note the relations

$$\begin{bmatrix} 2 \\ 0 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \text{ and } \begin{bmatrix} -1 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix},$$

which make it clear that each column is a linear combination of the two given vectors. Many other examples are possible.

Grading Rubric: 3 points for a correct matrix. 5 points for an additional explanation that all columns are linear combinations of the two given vectors.

Problem 2: Let us consider an $m \times n$ matrix R with block decomposition:

$$R = \begin{bmatrix} I & X \\ 0 & 0 \end{bmatrix} \tag{1}$$

where I is a (square) unit matrix. So in other words, R is in reduced row echelon form with all the pivot columns to the left of the free columns. Let r be the rank of R.

(1) What are the number of rows and columns of each of the 4 blocks in (1), in terms of m, n and r? *(10 points)*

(2) If $r = m$, find a right-inverse for R, i.e. a matrix Q such that $RQ = I_m$ *(5 points)*

(3) If $r = n$, find a left-inverse for R, i.e. a matrix Q such that $QR = I_n$ *(5 points)*

(4) If you are allowed to perform column operations on R (i.e. adding arbitrary multiples of any column to any other column), then what is the simplest form to which you can bring R? *(5 points)*

Solution:

(1) The I matrix is $r \times r$. This is because the rank of any matrix is equal to the number of non-zero rows in its reduced row echelon form, and the matrix R is already in its reduced row echelon form. The X matrix is $r \times (n-r)$. It has the same number r of rows as I, and the total number of columns in I and X adds up to n. The lower left 0 matrix is $(m-r) \times r$. It has the same number r of columns as I, and the total number of rows in I and this matrix adds up to m. The lower right 0 matrix is $(m-r) \times (n-r)$.

Grading rubric: 2 points for each of the blocks. The remaining 2 points should be given for a good explanation of the fact that the rank of R is the size of I.

(2) If $r = m$, then the matrix looks like

$$R = \begin{bmatrix} I & X \end{bmatrix},$$

with no zero blocks. A right inverse is given by the $n \times m$ matrix

$$Q = \begin{bmatrix} I \\ 0 \end{bmatrix}.$$

Grading rubric: 1 point if the number of rows and columns of the right inverse is correctly identified. Full credit if the answer is correct (no explanation needed).

(3) If $r = n$, then the matrix looks like

$$R = \begin{bmatrix} I \\ 0 \end{bmatrix},$$

with no X. A left inverse is given by the $n \times m$ matrix

$$\begin{bmatrix} I & 0 \end{bmatrix}$$

Grading rubric: 1 point if the number of rows and columns of the left inverse is correctly identified. Full credit if the answer is correct (no explanation needed).

(4) The simplest possible form is

$$\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

For $i \leq r$, we subtract multiples of the ith column of R to eliminate the ith row of X.

Grading rubric: 3 points for the correct answer. 5 points for the correct answer with a correct explanation.

Problem 3: The diagram below represents 5 nodes (represented by the circles) connected by 7 pieces of conducting wire (represented by the lines). The intensity of the current flowing through these pieces of wire is $x_1, ..., x_7$, in the direction of the arrow.

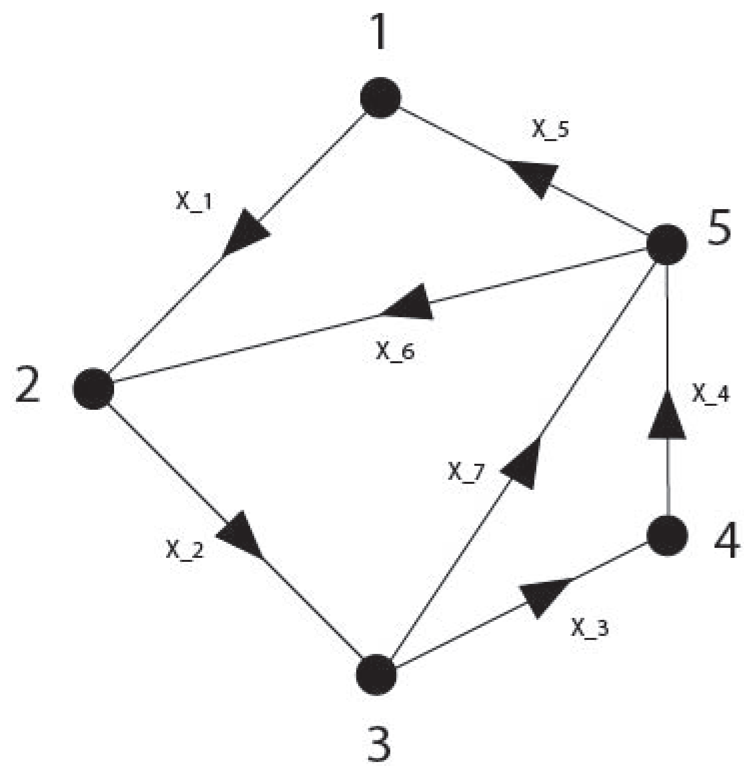
Kirchoff's first law says that, at every node, the incoming current should equal the outgoing current.

(1) Write down explicitly the incidence matrix of the diagram. By definition, this is the 5×7 matrix A whose entry at row i and column j is:

$$\begin{cases} 1 & \text{if the current on the } j\text{–th wire flows into node } i \\ -1 & \text{if the current on the } j\text{–th wire flows out of node } i \\ 0 & \text{if the } j\text{–th wire does not intersect node } i \end{cases}$$

(the j–th wire is the one denoted by the variable x_j in the diagram). *(5 points)*

(2) Express Kirchoff's first law as a linear algebra condition on the vector of currents $\begin{bmatrix} x_1 \\ ... \\ x_7 \end{bmatrix}$, which involves the incidence matrix A (justify). *(5 points)*



(3) By using the row reduced echelon form of A, find all possible vectors of currents $\begin{bmatrix} x_1 \\ ... \\ x_7 \end{bmatrix}$ which satisfy Kirchoff's law for the diagram above. *(10 points)*

Solution:
(1) The incidence diagram is

$$\begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}.$$

Grading rubric: -1 point for each incorrect entry. If the 1s and -1s are swapped, 3 points.

(2) The condition is just

$$A \begin{bmatrix} x_1 \\ ... \\ x_7 \end{bmatrix} = \begin{bmatrix} 0 \\ ... \\ 0 \end{bmatrix}.$$

Comparing first rows on both sides of this equation, for example, states that $-x_1 + x_5 = 0$, and this corresponds to the condition that incoming current equal outgoing current in node 1. Similarly, the ith row of the equation enforces Kirchoff's law at node i.

Grading rubric: 3 points for a correct equation. 5 points for the equation with a valid justification.

(3) The row reduced form is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the rank of this matrix is 4, but there are 7 columns, we may freely choose $7-4=3$ variables. In other words, x_5, x_6, and x_7 may be arbitary. However, Kirchoff's law constrains the other x_i to satisfy:

$$x_1 = x_5,$$

$$x_2 = x_5 + x_6,$$

$$x_3 = x_5 + x_6 - x_7, \text{ and}$$

$$x_4 = x_5 + x_6 - x_7.$$

Problem 4: (justify all your answers)

(1) If X is an invertible square matrix, what can you say about $C(X)$ and $N(X)$? *(10 points)*

(2) If $Y = \begin{bmatrix} A \\ B \end{bmatrix}$ is a block matrix, what is $N(Y)$ in terms of $N(A)$ and $N(B)$? *(5 points)*

(3) If $Z = \begin{bmatrix} A & B \end{bmatrix}$ is a block matrix, what is $C(Z)$ in terms of $C(A)$ and $C(B)$? *(5 points)*

Solution:

(1) Suppose X is $n \times n$. First, we claim that $N(X)$ consists only of the origin, so $N(X) = \{0\}$. To see this, suppose that $Xv = 0$. Then $X^{-1}Xv = X^{-1}0$, and so v must be 0. Next, we claim that $C(X)$ must be the entire space $\mathbb{R}^n$. Indeed, if v is any vector in $\mathbb{R}^n$, we can write $X(X^{-1}v) = v$, and so the entries of $X^{-1}v$ tell us what linear combination of columns of X produces v.

Grading rubric: 3 points for the correct statement about $N(X)$, 3 points for the correct statement about $C(X)$, and 2 additional points for each of the two explanations.

(2) $N(Y)$ is the intersection of $N(A)$ and $N(B)$. In other words, a vector v is in $N(Y)$ if and only if both $Av = 0$ and $Bv = 0$. To see this, note that the product Yv will just be

$$Yv = \begin{bmatrix} A \\ B \end{bmatrix} v = \begin{bmatrix} Av \\ Bv \end{bmatrix},$$

and this is clearly 0 if and only if $Av = Bv = 0$.

Grading rubric: 3 points for the correct answer. 5 points for the correct answer with explanation.

(3) $C(Z)$ is the sum of $C(A)$ and $C(B)$. By this, we mean that a vector v is in $C(Z)$ if and only if v can be written as a sum $v = w_1 + w_2$, where w_1 is in the column space of A and w_2 is in the column space of B. Indeed, the column space of Z consists of linear combinations of the columns of Z, and such a combination can be broken up into the sum of a linear combination of the columns of A and a linear combination of columns of B.

Grading rubric: 3 points for the correct answer. 5 points for the correct answer with explanation.

Problem 5:

(1) Compute the reduced row echelon form of the matrix:

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & -3 \\ 2 & 4 & 0 & -6 \end{bmatrix}$$

(all zero rows should be at the bottom of A). *(10 points)*

(2) Use the result of part (1) to find the full set of solutions to the equation:

$$A \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = 0$$

(10 points)

Solution:
(1)

$$\begin{bmatrix} 0 & 1 & -1 & -1 \\ 1 & 2 & 0 & -3 \\ 2 & 4 & 0 & -6 \end{bmatrix} \xrightarrow{swap} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & -1 \\ 2 & 4 & 0 & -6 \end{bmatrix} \xrightarrow{r_3 - 2r_1} \begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Grading rubric: 10 points for correct answer with steps shown. -5 points for no steps shown. -2 points if the final answer is wrong because of a calculation mistake. -2 points if the final answer has the row of 0s somewhere other than the bottom row.

(2) The solutions to a system of equations do not change under row operations, so we may as well solve the system

$$a + 2c - d = 0, \text{ and}$$
$$b - c - d = 0.$$

We can let d and c be arbitrary. From the second equation we learn that $b = c + d$. From the first equation we learn that $a = -2c + d$. The generic solution looks like

$$\begin{bmatrix} -2c + d \\ c + d \\ c \\ d \end{bmatrix}$$

Grading rubric: 5/10 points if it is realized that the set of solutions should be 2-dimensional. 8/10 points if the answer is only incorrect because of a calculation mistake.